Cobb-Douglas maximization with second-order conditions

Given the following utility function:

$$U(x_1, x_2) = Ax_1^{\alpha} x_2^{1-\alpha} \quad \text{with} \quad 0 < \alpha < 1,\ A > 0,\ x_1 > 0,\ x_2 > 0,$$

find the optimal consumption bundle if the consumer has the following constraint:

$$P_1 x_1 + P_2 x_2 = R \quad \text{with} \quad P_1 > 0,\ P_2 > 0,\ R > 0.$$

Show that the second-order conditions are verified.

Solution

Given:

$$U = Ax_1^{\alpha}x_2^{1-\alpha}$$

subject to the budget constraint:

$$P_1x_1 + P_2x_2 = R$$

The Lagrangian function is:

$$\mathcal{L} = Ax_1^{\alpha}x_2^{1-\alpha} + \lambda(R - P_1x_1 - P_2x_2)$$

First, we derive with respect to x_1:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha Ax_1^{\alpha-1}x_2^{1-\alpha} - \lambda P_1 = 0 \implies \lambda = \frac{\alpha Ax_1^{\alpha-1}x_2^{1-\alpha}}{P_1}$$

Then, we derive with respect to x_2:

$$\frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha)Ax_1^{\alpha}x_2^{-\alpha} - \lambda P_2 = 0 \implies \lambda = \frac{(1-\alpha)Ax_1^{\alpha}x_2^{-\alpha}}{P_2}$$

Equating the two values of λ:

$$\frac{\alpha Ax_1^{\alpha-1}x_2^{1-\alpha}}{P_1} = \frac{(1-\alpha)Ax_1^{\alpha}x_2^{-\alpha}}{P_2}$$

Simplifying:

$$\frac{\alpha x_2}{P_1} = \frac{(1-\alpha)x_1}{P_2}$$

Solving for x_2:

$$x_2 = \left(\frac{(1-\alpha)}{P_2} \cdot \frac{P_1}{\alpha}\right)x_1$$

Substituting into the budget constraint:

$$R = P_1x_1 + P_2x_2$$

$$R = P_1x_1 + P_2\left(\frac{(1-\alpha)}{P_2} \cdot \frac{P_1}{\alpha}\right)x_1$$

$$R = P_1x_1\left(1 + \frac{(1-\alpha)}{\alpha}\right)$$

$$R = P_1x_1\left(\frac{\alpha + (1-\alpha)}{\alpha}\right)$$

$$R = P_1x_1\left(\frac{1}{\alpha}\right)$$

$$x_1^* = \frac{R\alpha}{P_1}$$

Finally, substituting x_1^* into the equation for x_2:

$$x_2^* = \left(\frac{(1-\alpha)}{P_2} \cdot \frac{P_1}{\alpha}\right)\left(\frac{R\alpha}{P_1}\right)$$

$$x_2^* = \frac{R}{P_2}(1-\alpha)$$

Calculating the second derivatives:

$$\mathcal{L}''_{x_1x_1} = A\alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha}$$

$$\mathcal{L}''_{x_2x_2} = -\alpha(1-\alpha)Ax_1^{\alpha}x_2^{-\alpha-1}$$

$$\mathcal{L}''_{x_1x_2} = \mathcal{L}''_{x_2x_1} = \alpha(1-\alpha)Ax_1^{\alpha-1}x_2^{-\alpha}$$

And the first derivatives of the constraint to form the bordered Hessian:

$$g'_x = P_1$$

$$g'_y = P_2$$

We form the bordered Hessian:

$$|\bar{H}| = \begin{vmatrix} 0 & P_1 & P_2 \\ P_1 & A\alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)Ax_1^{\alpha-1}x_2^{-\alpha} \\ P_2 & \alpha(1-\alpha)Ax_1^{\alpha-1}x_2^{-\alpha} & -\alpha(1-\alpha)Ax_1^{\alpha}x_2^{-\alpha-1} \end{vmatrix}$$

$$|\bar{H}| = -P_1[-P_1\alpha(1-\alpha)Ax_1^{\alpha}x_2^{-\alpha-1} - P_2\alpha(1-\alpha)Ax_1^{\alpha-1}x_2^{-\alpha}] + P_2[P_1\alpha(1-\alpha)Ax_1^{\alpha-1}x_2^{-\alpha} - P_2A\alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha}]$$

Distributing the negative signs:

$$|\bar{H}| = P_1[P_1\alpha(1-\alpha)Ax_1^{\alpha}x_2^{-\alpha-1} + P_2\alpha(1-\alpha)Ax_1^{\alpha-1}x_2^{-\alpha}] + P_2[P_1\alpha(1-\alpha)Ax_1^{\alpha-1}x_2^{-\alpha} + P_2A\alpha(1-\alpha)x_1^{\alpha-2}x_2^{1-\alpha}]$$

We know that $0 < \alpha < 1$, so $1-\alpha > 0$. This results in all positive terms:

$$|\bar{H}| = \underbrace{P_1[P_1\alpha\underbrace{(1-\alpha)}_{+}Ax_1^{\alpha}x_2^{-\alpha-1} + P_2\alpha\underbrace{(1-\alpha)}_{+}Ax_1^{\alpha-1}x_2^{-\alpha}]}_{+} + \underbrace{P_2[P_1\alpha\underbrace{(1-\alpha)}_{+}Ax_1^{\alpha-1}x_2^{-\alpha} + P_2A\alpha\underbrace{(1-\alpha)_{+}}x_1^{\alpha-2}x_2^{1-\alpha}]}_{+}$$

Since we also know that $x_1^* > 0$ and $x_2^* > 0$, we can conclude that $|\bar{H}| > 0$ and since we have:

$$\mathcal{L}''_{x_1x_1} = A\alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} < 0$$

We have a maximum.